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(p, q, r)-Generations of the Smallest Conway Group Co_3

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An (l, m, n)-generated group G is a quotient group of the triangle group $T(l, m, n) = \langle x, y, z \mid x^l = y^m = z^n = xyz = 1 \rangle$. In this paper the authors continue the study on the (p, q, r)-generations of the sporadic simple groups, where p, q, r are distinct primes. The problem is resolved for the groups Co_3.

1. INTRODUCTION

A group G is said to be (l, m, n)*-generated* if $G = \langle x, y \rangle$, with $o(x) = l$, $o(y) = m$, and $o(xy) = n$. In this case G is a quotient group of the triangular group $T(l, m, n)$, and by the definition of the triangular group, G is also $(\pi(l), \pi(m), \pi(n))$-generated for any $\pi \in S_3$. We may therefore assume $l \leq m \leq n$. It is also well known that if G is an (l, m, n)-generated simple group, then $1/l + 1/m + 1/n < 1$(*cf*. [1]).

In a series of papers the authors established the $(2, 3, p)$--generations of the sporadic simple group F_{22} (*cf*. [11]) and the (p, q, r)-generations of the

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groups J_1, J_2, J_3, HS, and McL (*cf*. [7–10]). For more information and results on the 2-generations of finite simple groups and related topics, the reader is referred to [5] and [15–17]. This paper is devoted to the (p, q, r)-generations and its consequences for the Conway group Co_3, where p, q, r are distinct primes. We summarize our findings in the following theorem.

THEOREM 5.2. *The Conway group Co_3 is (p, q, r)-generated for all $p, q, r \in \{2, 3, 5, 7, 11, 23\}$ with $p < q < r$, except when $(p, q, r) = (2, 3, 5)$.*

The content of this paper will be organized as follows. In Section 2 we discuss techniques that are useful in resolving generation type questions of finite groups. As a consequence of the remark in the first paragraph, we only need ot consider the cases $r = 7, 11, 23$ for the group Co_3. We deal separately with each case in Sections 3, 4, and 5. For basic properties of the group Co_3 and information on its subgroups the reader is referred to [4, 6]. The "ATLAS of Finite Groups" [3] is an important reference and we adopt its notation for subgroups, conjugacy classes, etc. Computations were carried out with the aid of GAP [13] running on a Sun GX2 computer. The nX-complementary generations of the group Co_3 will be treated by the authors in a separate paper.

2. PRELIMINARY RESULTS

Throughout this paper we use the same notation as in [7–9], and the reader is encouraged to consult these papers for basic computational techniques. In particular, $\Delta(G) = \Delta_G(lX, mY, nZ)$ denotes the structure constant of G for the conjugacy classes lX, mY, nZ whose value is the cardinality of the set

$$\Gamma = \{(x, y) \in lX \times mY \mid xy = z\},$$

where z is a fixed element of the class nZ. Also, $\Delta^*(G) = \Delta^*_G(lX, mY, nZ)$ and $\Sigma(H_1 \cup \ldots \cup H_r)$ denote the number of pairs $(x, y) \in \Gamma$ such that $\langle x, y\rangle = G$ and $\langle x, y\rangle \leq H_i$ (for some $1 \leq i \leq r$), respectively. The number of $(x, y) \in \Gamma$ generating a subgroup H of G will be given by $\Sigma^*(H)$ and the centalizer of a representative of the conjugacy class lX we will denote by $C_G(lX)$. If $\Delta^*(G) < 0$, then we say G is (lX, mY, nZ)-generated and (lX, mY, nZ) is called a generating triple for G.

We will employ results that, in certain situations, will effectively establish non-generation. They include Scott's theorem (*cf*. [2, 14]), Ree's theorem (*cf*. [2, 12]), and Lemma 3.3 in [17]. The following result will be crucial in determining generating triples.

THEOREM 2.1. *Let G be a finite group and H a subgroup of G containing a fixed element x such that* $\gcd(o(x), [N_G(H){:}H]) = 1$. *Then the number h of conjugates of H containing x is $\chi_H(x)$, where χ_H is the permutation character of G with action on the conjugates on H. In particular,*

$$h = \sum_{i=1}^{m} \frac{|C_G(x)|}{|C_{N_G(h)}(x_i)|},$$

where $x_1 \dots x_m$ are representatives of the $N_G(H)$-conjugacy classes that fuse to the G-class $[x]_G$.

Proof. Let Ω be the set of all conjugates of the subgroup H. Then G acts (by conjugation) transitively on Ω and the point stabilizer of H is $N_G(H)$. Thus the permutation character of G with this action on Ω is $\chi_H = (1_{N_G(H)})^G$. By definition

$$\chi_H(x) = |\{H^g | (H^g)^x = H^g\}| = |\{H^g | x \in N_G(H^g)\}|$$

is the number of fix points of this action on Ω. Let $\bar{x}$ be the image of x under the natural homomorphism $N_G(H^g) \to N_G(H^g)/H^g$. Since $(o(x), [N_G(H^g){:}H^g]) = 1$, it follows that $o(\bar{x}) = 1$ and hence $x \in H^g$. Therefore $\chi_H(x) = |\{H^g | x \in H^g\}|$. On the other hand,

$$\chi_H(x) = \left(1_{N_G(H)}\right)^G(x) = \sum_{i=1}^{m} \frac{|C_G(x)|}{C_{N_G(H)}(x_i)|},$$

where $[x]_G \cap N_G(H) = \cup_{i=1}^{m} [x_i]_{N_G(H)}$. ∎

In the case where H is self-normalizing (for instance if H is a maximal subgroup of a simple group), the theorem reduces to the result used by Woldar in [17]. Whenever the permutation character of G on the conjugates of H is not known explicitly in terms of the irreducible characters of G, we use the fusion map of $N_G(H)$ into G to determine its value on the conjugacy classes. It is well known that $\Delta^*(G) = \Delta(G) - \Sigma(M_1 \cup \dots \cup M_s)$, where $\{M_1, \dots, M_s\}$ is the family of all maximal subgroups of G containing z. We also use the fusion maps in calculating $\Sigma(M_i)$, for $i = 1, \dots, s$. We list in Table I the partial fusion map of the maximal subgroups M of Co_3 (obtained from GAP) that we will use later. All the M-conjugacy classes with representatives of prime order are given. From the fusion maps we calculate the value of h (see Theorem 2.1), for $x \in M$ with $o(x) \geq 7$.

3. $(p, q, 7)$-GENERATION OF Co_3

The group Co_3 acts as a transitive *rank*-2 group on a set Ω of 276 points. The point stabilizer of this action is isomorphic to the group $McL{:}2$

TABLE I
Partial Fusion Maps into Co_3

$McL{:}2$-class	$2a$	$2b$	$3a$	$3b$	$5a$	$5b$	$7a$	$11a$	$11b$	
$\rightarrow Co_3$	$2A$	$2B$	$3A$	$3B$	$5A$	$5B$	$7A$	$11A$	$11B$	
h							3	1	1	
HS-class	$2a$	$2b$	$3a$	$5a$	$5b$	$5c$	$7a$	$11a$	$11b$	
$\rightarrow Co_3$	$2A$	$2B$	$3B$	$5A$	$5B$	$5B$	$7A$	$11A$	$11B$	
h							6	2	2	
$U_4(3){:}2^2$-class	$2a$	$2b$	$2c$	$2d$	$2e$	$3a$	$3b$	$3c$	$5a$	$7a$
$\rightarrow Co_3$	$2A$	$2A$	$2B$	$2B$	$2B$	$3A$	$3B$	$3B$	$5B$	$7A$
h										3
M_{23}-class	$2a$	$3a$	$5a$	$7a$	$7b$	$11a$	$11b$	$23a$	$23b$	
$\rightarrow Co_3$	$2A$	$3B$	$5B$	$7A$	$7A$	$11A$	$11B$	$23A$	$23B$	
h				3	3	2	2	1	1	
$3^5{:}(2 \times M_{11})$-class	$2a$	$2b$	$2c$	$3a$	$3b$	$3c$	$3d$	$5a$	$11a$	$11b$
$\rightarrow Co_3$	$2A$	$2B$	$2B$	$3A$	$3B$	$3B$	$3C$	$5B$	$11A$	$11B$
h									1	1
$2 \cdot S_6(2)$-class	$2a$	$2b$	$2c$	$3a$	$3b$	$3c$	$5a$	$7a$		
$\rightarrow Co_3$	$2A$	$2A$	$2B$	$3A$	$3A$	$3B$	$5A$	$7A$		
h								3		
$U_3(5){:}S_3$-class	$2a$	$2b$	$3a$	$3b$	$3c$	$5a$	$5b$	$7a$		
$\rightarrow Co_3$	$2A$	$2B$	$3B$	$3A$	$3C$	$5A$	$5B$	$7A$		
h								2		
$2^4 \cdot A_8$-class	$2a$	$2b$	$2c$	$3a$	$3b$	$5a$	$7a$	$7b$		
$\rightarrow Co_3$	$2A$	$2B$	$2A$	$3B$	$3B$	$5B$	$7A$	$7A$		
h							3	3		
$L_3(4){:}D_{12}$-class	$2a$	$2b$	$2c$	$2d$	$3a$	$3b$	$3c$	$5a$	$7a$	
$\rightarrow Co_3$	$2A$	$2B$	$2A$	$2B$	$3B$	$3B$	$3C$	$5B$	$7A$	
h									1	
$2 \times M_{12}$-class	$2a$	$2b$	$2c$	$2d$	$2e$	$3a$	$3b$	$5a$	$11a$	$11b$
$\rightarrow Co_3$	$2B$	$2A$	$2B$	$2B$	$2B$	$3B$	$3C$	$5B$	$11A$	$11B$
h									1	1

and the resulting permutation character is $\chi_{McL:2} = 1a + 275a$. The value of $\chi_{McL:2}$ on the conjugacy class pX, p a prime, will enable us to deduce the cycle type of elements in pX as a permutation of degree 276.

LEMMA 3.1. *The group Co_3 is $(2X, 3Y, 7A)$-generated, for $X \in \{A, B\}$ and $Y \in \{A, B, C\}$, if and only if the ordered pair $(X, Y) = (B, C)$.*

Proof. If the group Co_3 is (pX, qY, rZ)-generated, then an application of Ree's theorem. (*cf*. [12]) to Co_3 as a permutation group on 276 points

TABLE II
Cycle Type of a Representative in pX

Co_3-class	$2A$	$2B$	$3A$	$3B$	$3C$	$7A$	$11AB$
$\chi_{McL:2}(pX)$	36	12	6	15	0	3	1
Cycle type	$1^{36}2^{120}$	$1^{12}2^{132}$	$1^{6}3^{90}$	$1^{15}3^{87}$	3^{92}	$1^{3}7^{39}$	$1^{1}11^{25}$

implies that $c_1 + c_2 + c_3 \leq 278$, where c_1, c_2, and c_3 are the number of cycles of representatives in pX, qY, and rZ, respectively. From Table II we conclude that Ree's theorem is violated for all triples $(2X, 3Y, 7A)$, except when $X = B$ and $Y = C$.

For the triple $(2B, 3C, 7A)$ we observe from the fusion maps into Co_3 that if M is a maximal subgroup with non-empty intersection with the classes in this triple, then M is isomorphic to either $U_3(5){:}S_3$, $L_3(4){:}D_{12}$, or $S_3 \times L_2(8){:}3$. However, we easily calculate $\Sigma(M) = 0$ for all the above subgroups and hence $\Delta^*(Co_3) = \Delta(Co_3 = 504$, proving the result. ∎

LEMMA 3.2. *The group* Co_3 *is* $(2X, 5Y, 7A)$*-generated for* $X, Y \in \{A, B\}$, *if and only if the ordered pair* $(X, Y) \in \{(B, A), (B, B)\}$.

Proof. We calculate $\Delta_{Co_3}(2A, 5A, 7A) = 21 < |C_{Co_3}(7A)| = 42$ and non-generation of Co_3 by this triple follows from Lemma 3.3 in [17]. The group Co_3 acts on a 23-dimensional irreducible complex module V with

$$\dim\big(V/C_V(2A)\big) = 8, \qquad \dim\big(V/C_V(5B)\big) = 16, \qquad \dim\big(V/C_V(7A)\big) = 18.$$

But $8 + 16 + 18 < 46$, and hence, by Scott's theorem (*cf*. [14]), $(2A, 5B, 7A)$ is a non-generating triple of Co_3.

Next we consider the triple $(2B, 5A, 7A)$. The maximal subgroups of Co_3 with order divisible by 7 and non-empty intersection with the classes $2B$ and $5A$ are isomorphic to $McL{:}2$, HS, $2 \cdot S_6(2)$, and $U_3(5){:}S_3$. We calculate $\Delta(Co_3) = 1512$, $\Sigma(McL{:}2) = 0 = \Sigma(U_3(5){:}S_3), \Sigma(HS) = 48$, and $\Sigma(2 \cdot S_6(2)) = 98$. A fixed element of order 7 is contained in six conjugate subgroups of HS and in three conjugate copies of $2 \cdot S_6(2)$ (see Table I). Thus $\Delta^*(Co_3) \geq 1512 - 6(48) - 3(98) > 0$ and therefore $(2B, 5A, 7A)$ is a generating triple for Co_3.

Finally, we show that Co_3 is $(2B, 5B, 7A)$-generated. The maximal subgroups with non-empty intersection with the classes $2B$, $5B$, and $7A$ are, up to isomorphisms, $McL{:}2$, HS, $U_4(3){:}2^2$, $U_3(5){:}S_3$, $2^4 \cdot A_8$, and $L_3(4){:}D_{12}$. We calculate $\Delta(Co_3) = 7560$, $\Sigma(HS) = 532$, $\Sigma(2^4 \cdot A_8) = 56$, and $\Sigma(M) = 0$ for the remaining subgroups in the above list. Also a fixed element of order 7 is contained in three conjugate subgroups of $2^4 \cdot A_8$. Thus $\Delta^*(Co_3) \geq 7560 - 6(532) - 3(56) = 4200$, and the result follows. ∎

Let $\{H_1, \ldots, H_k\}$ be the set of all non-conjugate (lX, mY, nZ)-generated proper subgroups of G and let $z \in nZ$ be fixed. The number of pairs $(x, y) \in lX \times mY$ such that $xy = z$ and $\langle x, y \rangle \cong H_i$, for some i, is given by

$$\sum_{i=1}^{k} h_i \Sigma^*(H_i), \tag{1}$$

where h_i is the number of conjugates of H_i containing z. Without loss of generality, assume that $H_1, \ldots, H_{j-1}$ are subgroups of H_j, where $j \le k$. Now if $\langle x, y \rangle = H_i^g$, for some $g \in G$ and $i < j$, then $\langle x, y \rangle \le H_j^g$. Thus from the definition of $\Sigma(H_j)$ it follows that

$$\sum_{i=1}^{k} h_i \Sigma^*(H_i) \le h_j \Sigma(H_j) + \sum_{i=j+1}^{k} h_i \Sigma^*(H_i).$$

thus an understanding of the subgroup lattice of G will simplify the task of finding an upperbound for Eq. (1).

The group Co_3 acts transitively on the set Ω of conjugates of the subgroup $M \cong McL$. In [6] the lengths of the orbits of subgroups of Co_3 acting on Ω are determined. We shall use these orbit lengths to obtain information on the subgroup lattice of Co_3.

LEMMA 3.3. *The group Co_3 is $(3X, 5Y, 7A)$-generated, for all $X \in \{A, B, C\}$ and $Y \in \{A, B\}$.*

Proof. We will treat each triple separately.

Case $(3A, 5A, 7A)$. The maximal subgroups of Co_3 that have non-empty intersection with the classes $3A$, $5A$, and $7A$ are, up to isomorphisms, $McL{:}2$, $2 \cdot S_6(2)$, and $U_3(5){:}S_3$. We calculate $\Delta(Co_3) = 1680$, $\Sigma(McL{:}2) = 63$, $\Sigma(2 \cdot S_6(2)) = 70$, and $\Sigma(U_3(5){:}S_3) = 0$. From Table I it follows that $\Delta^*(Co_3) \ge 1680 - 3(63) - 3(70) = 1281$, and hence Co_3 is $(3A, 5A, 7A)$-generated.

Case $(3A, 5B, 7A)$. We calculate the structure constant $\Delta(Co_3) = 175518$. The maximal subgroups with non-empty intersection with the classes in this triple are isomorphic to $McL{:}2$, $U_4(3){:}2^2$, and $U_3(5){:}S_3$.We calculate

$$\Sigma(McL{:}2) = \Sigma(McL) = \Delta_{McL}(3a, 5b, 7x) = 644, \qquad x \in \{a, b\},$$

$$\Sigma(U_4(3){:}2^2) = \Sigma(U_4(3)) = \Delta_{U_4(3)}(3a, 5a, 7a) = 112,$$

$$\Sigma(U_3(5){:}S_3) = 0.$$

Clearly any $(3A, 5B, 7A)$-generated proper subgroup of Co_3 is contained in either McL or $U_4(3)$. From the list of maximal subgroups of $U_4(3)$ (*cf.*

[3]) we observe that, up to isomorphisms, only $L_3(4)$ and A_7 have order divisible by $3 \times 5 \times 7$. However, $\Sigma_{L_3(4)}(3a, 5a, 7a) = 0 = \Sigma_{A_7}(3a, 5a, 7a)$ and hence $\Sigma^*(U_4(3)) = \Delta_{U_4(3)}(3a, 5a, 7a) = 112$.

From Lemma 4.3 in [9] and the above argument it is clear that, up to isomorphisms, $U_4(3)$ is the only subgroup that contains $(3a, 5b, 7x)$-generated subgroups of McL. Furthermore, $\Sigma_{U_4(3)}(3a, 5b, 7x) = \Delta_{U_4(3)}(3a, 5a, 7a) = 112$ (the first part of the equality involves McL-classes and the second $U_4(3)$-classes). Therefore $\Sigma^*(McL) = 644 - 2(112) = 420$ (*cf*. [9]). We therefore conclude that McL and $U_4(3)$ are the only $(3A, 5B, 7A)$-generated proper subgroups of Co_3. It was shown by Finkelstein in [6] that Co_3 contains a unique conjugate class of subgroups isomorphic to McL and $U_4(3)$, respectively. Therefore

$$\begin{aligned}\Delta^*(Co_3) &= \Delta(Co_3) - 3\Sigma^*(McL) - 3\Sigma^*(U_4(3))\\ &= 1680 - 3(420) - 3(112) = 84,\end{aligned}$$

proving generation of Co_3 by this triple.

Case $(3B, 5B, 7A)$. We calculate the structure constant $\Delta^*(Co_3) = 175518$. From the fusion maps of the maximal subgroups into Co_3 we note that $McL{:}2$, HS, $U_4(3){:}2^2$, M_{23}, $U_3(5){:}S_3$, $2^4 \cdot A_8$, and $L_3(4){:}D_{12}$ are, up to isomorphisms, all the maximal subgroups that may contain $(3B, 5B, 7A)$-generated subgroups. We calculate

$$\begin{aligned}&\Sigma(McL{:}2) = \Sigma(McL) = 50400, \qquad \Sigma(HS) = 7280,\\ &\Sigma(U_4(3){:}2^2) = \Sigma(U_4(3)) = 9408, \qquad \Sigma(M_{23}) = 5124,\\ &\Sigma(U_3(5){:}S_3) = \Sigma(U_3(5)) = 420, \qquad \Sigma(2^4 \cdot A_8) = 420,\\ &\Sigma(L_3(4){:}D_{12}) = \Sigma(L_3(4)) = 882.\end{aligned}$$

Thus any $(3B, 5B, 7A)$-generated proper subgroup of Co_3 is contained in a subgroup isomorphic to McL, HS, $U_4(3)$, M_{23}, $U_3(5)$, $2^4 \cdot A_8$, or $L_3(4)$. By investigating the maximal subgroups of these groups and their fusions into Co_3, we find that the $(3B, 5B, 7A)$-generated proper subgroups of the above list are, up to isomorphisms, M_{22}, $2^4{:}A_7$, A_8, A_7, and (if possible) subgroups of $2^4 \cdot A_8$, other than $2^4{:}A_7$ and A_7.

We list in Table III the lengths of the orbits of the above subgroups acting on Ω. In this table m^n denotes n orbits of length m. If H is any subgroup of Co_3 fixing at least one point $M' \in \Omega$, then $H \leq G_{M'} \cong McL{:}2$. Thus it follows from Table III that any McL, $U_4(3)$, M_{22}, $U_3(5)$ (one fix point on Ω), $2^4{:}A_7$, $L_3(4)$ (both classes), and A_7 (both classes) subgroup is contained in some $McL{:}2$ subgroup of Co_3. It is shown in [6] that Co_3 contains a unique conjugate class for each of the remaining subgroups in Table III.

TABLE III
Action of H on Ω

H	Length of Ω-orbits	$N_{Co_3}(H)$
McL	$[1, 275]$	$McL{:}2$
HS	$[100, 176]$	HS
$U_4(3)$	$[1^2, 112, 162]$	$U_4(3){:}2^2$
M_{23}	$[23, 253]$	M_{23}
$U_3(5)$	$[50^3, 126]$	$U_3(5){:}S_3$
$U_3(5)$	$[1, 50^2, 175]$	$U_3(5){:}2$
M_{22}	$[1, 22, 77, 176]$	M_{22}
$2^4 \cdot A_8$	$[8, 128, 140]$	$2^4 \cdot A_8$
$2^4{:}A_7$	$[1, 7, 16, 112, 140]$	
$L_3(4)$	$[1^3, 56^3, 105]$	$L_3(4){:}D_{12}$
$L_3(4)$	$[1^2, 21^2, 56^2, 120]$	
A_8	$[8, 15^2, 70, 168]$	S_8
A_7	$[1, 7, 15^2, 35^2, 42, 126]$	S_7
A_7	$[1^2, 7, 15, 35, 42, 70, 105]$	A_7

It therefore follows from Theorem 2.1 that the number of pairs $(x, y) \in 3B \times 5B$, with $xy = z$ a fixed element in $7A$ and $\langle x, y\rangle < Co_3$, is at most

$$3\Sigma(McL{:}2) + 6\Sigma^*(HS) + 3\Sigma^*(M_{23}) + 2\Sigma^*(U_3(5)) + 6\Sigma^*(A_8) + 3\Sigma(2^4 \cdot A_8). \tag{2}$$

We now proceed by finding an upperbound for the above equation. The groups A_7 and $L_3(4)$ contain no proper subgroups with order divisible by $3 \times 5 \times 7$ and hence $\Sigma^*(A_7) = \Sigma(A_7) = 63$ and $\Sigma^*(L_3(4)) = \Sigma(L_3(4)) = 882$. Up to isomorphisms, A_7 is the only subgroup of A_8 that admits $(3B, 5B, 7A)$-generation. Also a fixed element of order 7 is contained in a unique conjugate of an A_7 subgroup in A_8. Thus $\Sigma^*(A_8) = \Sigma(A_8) - \Sigma^*(A_7) = 84 - 63 = 21$.

For the group $U_3(5)$ we have

$$\Sigma(U_3(5)) = \Delta_{U_3(5)}(3a, 5b, 7x) + \Delta_{U_3(5)}(3a, 5c, 7x) + \Delta_{U_3(5)}(3a, 5d, 7x),$$

where $x \in \{a, b\}$. We calculate $\Delta_{U_3(5)}(3a, 5y, 7x) = 140$, where $y \in \{b, c, d\}$. Also the maximal subgroups of $U_3(5)$ with order divisible by $3 \times 5 \times 7$ are isomorphic to A_7 (three non-conjugate types). The fusion map of A_7 into $U_3(5)$ yields

$$3a \to 3a \;\; 3b \to 3a \;\; 5a \to 5y \;\; 7a \to 7a \;\; 7b \to 7b,$$

where $y = b, c, d$ if A_7 is of conjugate type (i), (ii), (iii), respectively. Also a fixed element of order 7 is contained in a unique A_7 subgroup of $U_3(5)$. Thus $\Delta^*_{U_3(5)}(3a, 5y, 7x) = 77$ and hence $\Sigma^*(U_3(5)) = 231$.

Next we consider the groups M_{22} and M_{23}. We note $\Sigma(M_{22}) = \Delta_{M_{22}}(3a, 5a, 7x)$, $x \in \{a, b\}$. The $(3a, 5a, 7x)$-generated subgroups of M_{22} are isomorphic to $L_3(4)$ and A_7 (two non-conjugate copies). Using Theorem 2.1 we obtain $\Sigma^*(M_{22}) = 2464 - 882 - 2(63) = 1456$. The $(3B, 5B, 7A)$-generated maximal subgroups of M_{23} are isomorphic to M_{22}, $L_3(4){:}2_2$, $2^4{:}A_7$, and A_8. From the previous arguments it follows that if H is a $(3B, 5B, 7A)$-generated proper subgroup of M_{23}, then H is isomorphic to either A_7, A_8, $2^4{:}A_7$, $L_3(4)$, M_{22}, or $H \leq 2^4{:}A_7$. We calculate $\Sigma(2^4{:}A_7) = 336$. Now $2^4{:}A_7$ contains a subgroup isomorphic to A_7, and from Theorem 2.1 we have

$$\Sigma^*(M_{23}) \leq \Sigma(M_{23}) - 2\Sigma^*(M_{22}) - 2\Sigma^*\big(L_3(4)\big) - 2\Sigma^*(A_8)$$
$$-\Sigma\big(2^4{:}A_7\big) = 70.$$

For the group HS we have $\Sigma(HS) = \Delta_{HS}(3a, 5b, 7a) + \Delta_{HS}(3a, 5c, 7a)$. From Lemma 2.4 in [9], it follows immediately that

$$\Delta_{HS}(3a, 5b, 7a) \leq 560 - 2(77) - 2(63) = 260,$$
$$\Delta_{HS}(3a, 5c, 7a) \leq 6720 - 2(1456) - 2(77) - 882 - 2(63) = 2646$$

and hence $\Sigma^*(HS) \leq 2906$.

Thus an upperbound for Eq. (2) is 170694. The $(3B, 5B, 7A)$-generation of Co_3 follows from $\Delta(Co_3) = 175518 > 170694$.

Case $(3C, 5A, 7A)$. We calculate $\Delta(Co_3) = 85428$. Up to isomorphisms, $U_3(5){:}S_3$ is the only maximal subgroup of Co_3 with non-empty intersection with the classes of this triple. However, $\Sigma(U_3(5){:}S_3 = 0$ so that $(3C, 5A, 7A)$ is a generating triple for Co_3.

Case $(3C, 5B, 7A)$. The maximal subgroups that contain possible $(3C, 5B, 7A)$-generated subgroups are isomorphic to $U_3(5){:}S_3$ and $L_3(4){:}D_{12}$. However, $\Sigma(U_3(5){:}S_3) = 0 = \Sigma(L_3(4){:}D_{12})$ and hence $\Delta^*(Co_3) = \Delta(Co_3) = 296136$, proving $(3C, 5B, 7A)$-generation of Co_3. ∎

4. $(p, q, 11)$-GENERATION OF Co_3

In this section we need only to consider the maximal subgroups of Co_3 with order divisible by 11. They are, up to isomorphisms, $McL{:}2$, HS, M_{23}, $3^5(2 \times M_{11})$, and $2 \times M_{12}$.

LEMMA 4.1. *The group Co_3 is $(2X, 3Y, 11Z)$-generated for $X, Z \in \{A, B\}$, and $Y \in \{A, B, C\}$, if and only if the ordered pair $(X, Y) = (B, C)$.*

Proof. An application of Ree's theorem the representatives of the classes $2A$, $3B$, and $11Z$ (*cf*. Table II) establishes that Co_3 is not $(2A, 3B, 11Z)$-generated. The action of Co_3 on the 23-dimensional irreducible complex module V yields

$$\dim(V/C_V(2A)) = 8, \quad \dim(V/C_V(2B)) = 12, \quad \dim(V/C_V(3B)) = 18,$$
$$\dim(V/C_V(3C)) = 12, \quad \dim(V/C_V(11AB)) = 20.$$

Thus the triples $(2A, 3C, 11Z)$ and $(2B, 3B, 11Z)$ violate Scott's theorem, resulting in the non-generation of Co_3 by these triples. Next we calculate the structure constants $\Delta_{Co_3}(2A, 3A, 11Z) = 0 = \Delta_{Co_3}(2B, 3A, 11Z)$ and non-generation by these triples is immediate.

Finally, we calculate $\Delta_{Co_3}(2B, 3C, 11Z) = 671$. The maximal subgroups of Co_3 that may contain $(2B, 3C, 11Z)$-generated subgroups are isomophic to $3^5{:}(2 \times M_{11})$ and $2 \times M_{12}$. Also $\Sigma(3^5{:}(2 \times M_{11})) = 0$ and $\Sigma(2 \times M_{12}) = 11$. From Table I we conclude $\Delta^*(Co_3) = \Delta(Co_3) - \Sigma(2 \times M_{12}) = 660$, proving the result. ∎

LEMMA 4.2. *The group Co_3 is $(2X, 5Y, 11Z)$-generated, for all $X, Y, Z \in \{A, B\}$, except when $(2X, 5Y, 11Z) = (2A, 5A, 11Z)$.*

Proof. We treat the four cases separately.

Case $(2A, 5A, 11Z)$. The structure constant $\Delta\ (Co_3) = 44$. From the fusion maps into Co_3 we note that the $(2A, 5A, 11Z)$-generated proper subgroups are contained in the maximal subgroups isomorphic to $McL{:}2$ or HS. Also $\Sigma(McL{:}2) = \Sigma(McL) = 22$ and $\Sigma(HS) = 11$. It follows from Lemmas 2.5 and 4.5 in [9] that no proper subgroup of McL or HS is $(2A, 5A, 11Z)$-generated. Thus from the fusion maps we have

$$\Delta^*(Co_3) = \Delta(Co_3) - \Sigma^*(McL) - 2\Sigma^*(HS) = 0,$$

proving non-generation of Co_3 by this triple.

Case $(2A, 5B, 11Z)$. Every maximal subgroup with order divisible by 11 has non-empty intersection with each of the classes $2A$, $5B$, and $11Z$. From the structure constants we calculate

$$\Sigma(McL{:}2) = \Sigma(McL) = 715, \quad \Sigma(HS) = 242, \quad \Sigma(M_{23}) = 235,$$
$$\Sigma(3^5{:}(2 \times M_{11})) = 99, \quad \Sigma(2 \times M_{12}) = \Sigma(M_{12}) = 55.$$

Using the ATLAS and subgroup fusions into Co_3, we identify all the possible $(2A, 5B, 11Z)$-generated proper subgroups of Co_3, up to isomophisms. They are McL, HS, M_{23}, M_{22}, M_{12}, M_{11}, $L_2(11)$, and subgroups of $3^5{:}(2 \times M_{11})$. In [6] it is shown that Co_3 has one conjugate class of M_{23}, M_{22}, M_{12}, M_{11}, and $L_2(11)$ subgroups respectively. Furthermore, since $3^5{:}(2 \times M_{11})$ contains M_{11} and $L_2(11)$ subgroups, every M_{11} and $L_2(11)$ subgroup of Co_3 is contained in some conjugate copy of a $3^5{:}(2 \times M_{11})$ subgroup. From Theorem 2.1, it follows that the number of pairs $(x, y) \in 2A \times 5B$, with $xy = z \in 11Z$ a fixed element and $\langle x, y \rangle < Co_3$, is at most

$$\Sigma(McL) + 2\Sigma^*(HS) + 2\Sigma^*(M_{23}) + \Sigma(M_{12}) + \Sigma\big(3^5{:}(2 \times M_{11})\big). \quad (3)$$

None of the subgroups of $L_2(11)$ has order divisible by $2 \times 5 \times 11$ and hence $\Sigma^*(L_2(11)) = \Sigma(L_2(11)) = 22$. Up to isomorphisms, $L_2(11)$ is the only proper subgroup of M_{11} that is $(2A, 5B, 11Z)$-generated and a fixed element of order 11 is contained in a unique $L_2(11)$ subgroup of M_{11}. Thus $\Sigma^*(M_{11}) = \Sigma(M_{11}) - \Sigma^*(L_2(11)) = 11$. Similarly, $\Sigma^*(M_{22}) = \Sigma(M_{22}) - \Sigma^*(L_2(11)) = 176 - 22 = 154$.

The only $(2A, 5B, 11Z)$-generated proper subgroups of each of the groups McL, HS, and M_{23} are isomorpic to M_{22}, M_{11}, and $L_2(11)$. A fixed element of order 11 (in M_{23}) is contained in a unique conjugate of an M_{22}, M_{11}, and $L_2(11)$ subgroup, respectively. Thus $\Sigma^*(M_{23}) = 253 - 154 - 11 - 22 = 66$. From [9] it follows immediately that $\Sigma^*(HS) = 50$ and $\Sigma^*(McL) = 374$. Thus from Eq. (3) an upperbound for the number of pairs from $2A \times 5B$ that produce $(2A, 5B, 11Z)$-generated proper subgroups is 762. The $(2A, 5B, 11Z)$-generation of Co_3 follows since $\Delta(Co_3) = 1023 > 762$.

Case $(2B, 5A, 11Z)$. We calculate $\Delta(Co_3) = 2068$. Any maximal subgroup with non-empty intersection with the classes $2B$, $5A$, and $11Z$ is isomorphic to $McL{:}2$ or HS. Furthermore, $\Sigma(McL{:}2) = 0$, $\Sigma(HS) = 33$, and therefore $\Delta^*(Co_3) \geq 2068 - 2(33) = 2002$, proving the generation of Co_3 by this triple.

Case $(2B, 5B, 11Z)$. The structure constant $\Delta(Co_3) = 7513$. We observe from Table I that the groups isomorphic to $McL{:}2$, HS, $3^5{:}(2 \times M_{11})$, and $2 \times M_{12}$ are the maximal subgroups of Co_3 that may contain $(2B, 5B, 11Z)$-generated subgroups. We calculate $\Sigma(McL{:}2) = 0$, $\Sigma(HS) = 638$, $\Sigma(3^5{:}(2 \times M_{11})) = 0$, and $\Sigma(2 \times M_{12}) = 33$. Thus $\Delta^*(Co_3) \geq 7513 - 2(638) - 33 > 0$, proving that $(2B, 5B, 11Z)$ is a generating triple of Co_3. ∎

LEMMA 4.3. *The group Co_3 is $(2X, 7A, 11Y)$-generated, for all X, $Y \in \{A, B\}$.*

Proof. *Case* $(2A, 7A, 11Y)$. The structure constant $\Delta(Co_3) = 6622$. The $(2A, 7A, 11Y)$-generated proper subgroups of Co_3 are contained in the maximal subgroups isomorphic to $McL{:}2$, HS, and M_{23}. We also calculate $\Sigma(McL{:}2) = 3168$, $\Sigma(HS) = 825$, and $\Sigma(M_{23}) = 616$. From Table I we conclude

$$\Delta^*(Co_3) \geq 6622 - 3168 - 2(825) - 2(616) > 0,$$

and generation of Co_3 by this triple follows.

Case $(2B, 7A, 11Y)$. Up to isomorphisms, $McL{:}2$ and HS are the only maximal subgroups that may admit $(2B, 7A, 11Y)$-generated subgroups. Also $\Delta(Co_3) = 57266$, $\Sigma(McL{:}2) = 0$, $\Sigma(HS) = 2211$, and hence $\Delta(Co_3) \geq 52844$, proving the result. ∎

LEMMA 4.4. *The group Co_3 is $(3X, 5Y, 11Z)$-generated, for all $X \in \{A, B, C\}$ and Y, $Z \in \{A, B\}$.*

Proof. *Case* $(3Z, 5Y, 11Z)$. The maximal subgroups of Co_3 with order divisible by 11 and non-empty intersection with the class $3A$ are isomorphic to $McL{:}2$ and $3^5{:}(2 \times M_{11})$. Furthermore, a $3^5{:}(2 \times M_{11})$ subgroup does not meet the class $5A$ and hence

$$\begin{aligned}\Delta^*_{Co_3}(3A, 5A, 11Z) &\geq \Delta_{Co_3}(3A, 5A, 11Z) - \Sigma_{McL:2}(3A, 5A, 11Z)\\ &= 1496 - 44\end{aligned}$$

and

$$\begin{aligned}\Delta^*_{Co_3}(3A, 5B, 11Z) &\geq \Delta_{Co_3}(3A, 5B, 11Z) - \Sigma_{McL:2}(3A, 5B, 11Z)\\ &\quad -\Sigma_{3^5:(2\times M_{11})}(3A, 5B, 11Z) = 1232,\end{aligned}$$

proving generation of Co_3 by these triples.

Case $(3B, 5A, 11Z)$. The $(3B, 5A, 11Z)$-generated proper subgroups of Co_3 are contained in the maximal subgroups isomorphic to $McL{:}2$ and HS. We calculate $\Delta(Co_3) = 6380$, $\Sigma(McL{:}2) = 1122$, $\Sigma(HS) = 244$, and hence $\Delta^*(Co_3) \geq 4770$, proving generation.

Case $(3B, 5B, 11Z)$. All maximal subgroups with order divisible by 11 have non-empty intersection with all the classes in the triple. Our calculations yield

$$\Delta^*(Co_3) \geq 92070 - 34485 - 2(5313) - 2(3795) - 891 - 198 = 38280,$$

proving generation of Co_3 by the triple $(3B, 5B, 11Z)$.

Case $(3C, 5Y, 11Z)$. The maximal subgroups of Co_3 with order divisible by 11 and non-empty intersection with the class $3C$ are up to isomorphism, $3^5{:}(2 \times M_{11})$ and $2 \times M_{12}$. However, the class $5A$ does not meet either of these subgroups. Since the structure constant $\Delta_{Co_3}(3C, 5A, 11Z) = 76472$, the $(3C, 5A, 11Z)$-generation of Co_3 is immediate. Next, $\Delta_{Co_3}(3C, 5B, 11Z) = 323081$, $\Sigma_{3^5:(2\times M_{11})}(3C, 5B, 11Z) = 1782$, $\Sigma_{2\times M_{12}}(3C, 5B, 11Z) = 253$. Thus $\Delta^*_{Co_3}(3C, 5B, 11Z) \geq 321046$ and the generation of Co_3 by this triple follows. ∎

LEMMA 4.5. *The group Co_3 is $(3X, 7A, 11Y)$-generated for all $X \in \{A, B, C\}$ and $Y \in \{A, B\}$.*

Proof. The maximal subgroups of Co_3 with order divisible by $3 \times 7 \times 11$ are, up to isomorphisms, $McL{:}2$, HS, and M_{23}. The subgroups HS and M_{23} have empty intersection with the class $3A$ and therefore

$$\begin{aligned}\Delta^*_{Co_3}(3A, 7A, 11Y) &= \Delta_{Co_3}(3A, 7A, 11Y) - \Sigma_{McL:2}(3A, 7A, 11Y)\\ &= 22000 - 4356 > 0.\end{aligned}$$

Next we calculate

$$\Delta_{Co_3}(3B, 7A, 11Y) = 580800, \qquad \Sigma_{McL:2}(3B, 7A, 11Y) = 132264,$$
$$\Sigma_{HS}(3B, 7A, 11Y) = 17622, \qquad \Sigma_{M_{23}}(3B, 7A, 11Y) = 8272,$$

so that $\Delta^*_{Co_3}(3B, 7A, 11Y) \geq 396648$. Finally, the maximal subgroups isomorphic to $McL{:}2$, HS, and M_{23} do not meet the class $3C$ and hence $\Delta^*_{Co_3}(3C, 7A, 11Y) = \Delta_{Co_3}(3C, 7A, 11Y) = 2374614$, proving the result. ∎

LEMMA 4.6. *The group Co_3 is $(5X, 7A, 11Y)$-generated, for all $X, Y \in \{A, B\}$.*

Proof. The maximal subgroups that may contain $(5, 7, 11)$-generated subgroups are isomorphic to $McL{:}2$, HS, and M_{23}. For the triple $(5A, 7A, 11Y)$ we have $5A \cap M_{23} = \emptyset$, $\Delta(Co_3) = 6498712$, $\Sigma(McL{:}2) = 171072$, and $\Sigma(HS) = 12672$ so that $\Delta^*(Co_3) \geq 6302296$. For the remaining case $(5B, 7A, 11Y)$, we calculate $\Delta(Co_3) = 49618756$, $\Sigma(McL{:}2) = 5132160$, $\Sigma(HS) = 274593$, and $\Sigma(M_{23}) = 97192$ and hence $\Delta^*(Co_3) > 0$, and the result follows. ∎

TABLE IV
Structure Constants of Co_3

pX	$3A$	$3B$	$3C$	$5A$	$5B$	$7A$	$11AB$
$\Delta_{Co_3}(2A, pX, 23Y)$	0	0	46	115	276	3197	7728
$\Delta_{Co_3}(2B, pX, 23Y)$	0	46	736	1955	6716	56971	120796
$\Delta_{Co_3}(3A, pX, 23Y)$	–	23	529	1380	3818	33350	66700
$\Delta_{Co_3}(3B, pX, 23Y)$	–	–	3542	10166	44160	361284	769350
$\Delta_{Co_3}(3C, pX, 23Y)$	–	–	–	70219	376372	2635317	4926278
$\Delta_{Co_3}(5A, pX, 23Y)$	–	–	–	–	817476	7893692	14954232
$\Delta_{Co_3}(5B, pX, 23Y)$	–	–	–	–	1106346	37913246	75202410
$\Delta_{Co_3}(7A, pX, 23Y)$	–	–	–	–	–	–	536538388

5. $(p, q, 23)$-GENERATION OF Co_3 AND MAIN RESULTS

The maximal subgroups of Co_3 containing elements of order 23 are isomorphic to M_{23}. It is evident from Table I that a fixed element of order 23 is contained in a unique conjugate of a M_{23} subgroup and such a subgroup has empty intersection with the classes $2B$, $3A$, $3C$, and $5A$. Thus whenever a triple $(pX, qY, 23Z)$ includes at least one of these classes then $\Delta^*(Co_3) = \Delta(Co_3)$. Moreover, if this triple contains none of these classes, then $\Delta^*(Co_3) = \Delta(Co_3) - \Sigma(M_{23})$.

LEMMA 5.1. *The group Co_3 is $(pX, qY, 23Z)$-generated, for primes $p \leq q$ and $pX \neq qY$, if and only if the ordered pair $(pX, qY) \notin \{(2A, 3A), (2A, 3B), (2B, 3A)\}$.*

Proof. The result is immediate from the above remarks and Tables IV and V. ∎

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We summarize the main results in the following theorems.

THEOREM 5.2. *The Conway group Co_3 is (p, q, r)-generated for all $p, q, r \in \{2, 3, 5, 7, 11, 23\}$ with $p < q < r$, except when $(p, q, r) = (2, 3, 5)$.*

TABLE V
Struture Constants $\Sigma(M_{23})$

pX	$5B$	$7A$	$11X$
$\Sigma_{M_{23}}(2A, pX, 23Y)$	138	368	391
$\Sigma_{M_{23}}(3B, pX, 23Y)$	2438	6624	5129
$\Sigma_{M_{23}}(5B, pX, 23Y)$	–	88320	61893
$\Sigma_{M_{23}}(7A, pX, 23Y)$	–	–	135424

Proof. This follows from the above lemmas and the fact that the triangular group $T(2, 3, 5) \cong A_5$. ∎

COROLLARY 5.3. *The Conway group Co_3 is (pX, pX, qY)-generated, for all $pX \in \{3C, 5A, 5B, 7A, 11A, 11B\}$ and $qY \in \{7A, 11A, 11B, 23A, 23B\}$ with $p < q$ as well as $(pX, pX, qY) = (3B, 3B, 23X)$.*

Proof. The result follows immediately from an application of Lemma 2 in [2] to Lemmas 3.1, 3.2, 4.1, 4.2, 4.3, and 5.1. ∎

REFERENCES

1.. M. D. E. Conder, Some results on quotients of triangle groups, *Bull. Austral. Math. Soc.* **30** (1984), 73–90.
2. M. D. E. Conder, R. A. Wilson, and A. J. Woldar, The symmetric genus of sporadic groups, *Proc. Amer. Math. Soc.* **116** (1992), 653–663.
3. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Wilson, "Atlas of Finite Groups," Oxford Univ. Press (Clarendon), Oxford, 1985.
4. R. T. Curtis, On subgroups of $\cdot O$. I. Lattice stabilizer, *J. Algebra* **27** (1973), 549–573.
5. L. Di Martino and C. Tamburini, 2-Generation of finite simple groups and some related topics, in "Generators and Relations in Groups and Geometry" (A. Barlotti *et al.*, Eds.), pp. 195–233, Kluwer Academic, New York, 1991.
6. L. Finkelstein, The Maximal subgroups of Conway's Group C_3 and Mclauglin's Group, *J. Algebra* **25** (1973), 58–89.
7. S. Ganief and J. Moori, $(2, 3, t)$-Generations for the Janko group J_3, *Comm. Algebra*, **23**, No. 12 (1995), 4427–4437.
8. S. Ganief and J. Moori, *rX*-Complementary generations of the Janko groups J_1, J_2 and J_3, *Comm. Algebra*, **24**, No. 3 (1996), 809–820.
9. S. Ganief and J. Moori, (p, q, r)-Generations and *nX*-complementary generations of the Sporadic groups *HS* and *McL*, *J. Algebra* **187** (1997).
10. J. Moori, (p, q, r)-Generations for the Janko groups J_1 and J_2, *Nova J. Algebra Geom.* **2** No. 3 (1993), 277–285.
11. J. Moori, $(2, 3, p)$-Generations for the Fischer group F_{22}, *Comm. Algebra* **22**, No. 11 (1994), 4597–4610.
12. R. Ree, A theorem on permutations, *J. Combin. Theory Ser. A* **10** (1971), 174–175.
13. M. Schönert *et al.* "GAP—Groups, Algorithms and Programming," Lehrstul D Für Mathematik, RWTH, Aachen, 1992.
14. L. L. Scott, Matrices and Cohomology, *Ann. of Math.* (3) **105** (1977), 473–492.
15. A. J. Woldar, On Hurwitz generation and genus actions of sporadic groups, *Illinois J. Math.* **33**, No. 3 (1989), 416–437.
16. A. J. Woldar, Sporadic simple groups which are Hurwitz, *J. Algebra* **144** (1991), 443–450.
17. A. J. Woldar, Representing M_{11}, M_{12}, M_{22} and M_{23} on surfaces of least genus, *Comm. Algebra* **18** (1990) 15–86.